AN UNSATURATED GENERIC STRUCTURE (Model theoretic aspects of the notion of independence and dimension)

IKEDA, KOICHIRO; KIKYO, HIROTAKA

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AN UNSATURATED GENERIC STRUCTURE

法政大学・経営　池田宏一郎 (KOICHIRO IKEDA)
FACULTY OF BUSINESS ADMINISTRATION, HOSEI UNIVERSITY

神戸大学・情報　桔梗宏孝 (HIROTAKA KIKYO)
DEPARTMENT OF INFORMATICS, KOBE UNIVERSITY

ABSTRACT. We construct an *ab initio* generic structure for a predimension function with a positive rational coefficient strictly less than 1 which is unsaturated and has a non-ω-stable theory. Superstability of the theory will be discussed in a sequel paper.

1. INTRODUCTION

We consider graph structures. A graph structure has one binary relation as a first order structure. $X \subseteq_{\text{fin}} Y$ means that X is a finite subset of Y.

For a graph structure A, let

$$\delta_\alpha(A) = |A| - \alpha e(A).$$

Here, α is a rational number such that $0 < \alpha < 1$, $|A|$ the number of points in A, and $e(A)$ the number of edges in A. $\delta_\alpha(A)$ is called a *predimension function.*

Suppose $A \subseteq_{\text{fin}} B$ (substructure = induced subgraph).

$A \leq B$ (A is a *strong substructure* of B or A is *closed* in B) if

$$A \subseteq X \subseteq_{\text{fin}} B \Rightarrow \delta_\alpha(A) \leq \delta_\alpha(X).$$

In this case, if $A = \{a\}$ (a singleton) then a is called a *closed point* in B.

We say that $A \leq B$ is *minimal* if $A \leq B$, $A \neq B$, and $A \leq X \leq B$ implies $X = A$ or $X = B$.

With this notation, let

$$\mathbf{K}_\alpha = \{A : \text{finite} : A \geq \emptyset\}.$$

Definition 1.1. Suppose $\mathbf{K} \subseteq \mathbf{K}_\alpha$. A countable graph M is a *generic structure* of $\mathbf{K}$ if

- $A \subseteq_{\text{fin}} M \Rightarrow$ there exists B such that $A \subseteq B \subseteq_{\text{fin}} M$ and $B \leq M$;
- $A \subset_{\text{fin}} M \Rightarrow A \in \mathbf{K}$;
- for any A, $B \in \mathbf{K}$,

$$\begin{tikzcd}
B \arrow[dr, dashed, "\leq"] & \\
A \arrow[u, "\leq"] \arrow[r, "\leq"'] \arrow[ur, phantom, "\circlearrowleft"] & M
\end{tikzcd}$$

Definition 1.2. A class $\mathbf{K}$ has the *amalgamation property* (AP, in short) if for any $A, B, C \in \mathbf{K}$,

$$\begin{tikzcd}
 & B \arrow[dr, dashed, "\le"] & \\
A \arrow[ur, "\le"] \arrow[dr, "\le"'] \arrow[rr, phantom, "\circlearrowleft"] & & {}^{\exists}D \in \mathbf{K} \\
 & C \arrow[ur, dashed, "\le"'] &
\end{tikzcd}$$

Fact 1.3. *Suppose* $\mathbf{K} \subseteq \mathbf{K}_\alpha$*,*

(1) $\emptyset \in \mathbf{K}$*,*
(2) $\mathbf{K}$ *has the AP, and*
(3) $A \subset B \in \mathbf{K}$ *implies* $A \in \mathbf{K}$.

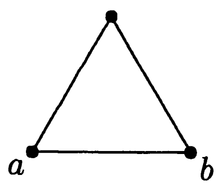
Then $\mathbf{K}$ *has a generic structure.*

Definition 1.4. Suppose $\mathbf{K} \subseteq \mathbf{K}_\alpha$. $\mathbf{K}$ has *thrifty amalgamation* if whenever $A \le B$ is minimal, $A \le C$ with $A, B, C \in \mathbf{K}$ then either $B \oplus_A C \in \mathbf{K}$ or there is a strong embedding of B into C over A.

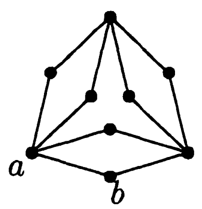
2. An Amalgamation Class

Definition 2.1. A graph A is a *minimal* 1*-component* (in $\mathbf{K}_\alpha$) if $|A| \ge 2$, $\delta_\alpha(A) = 1$, and $\delta_\alpha(X) > 1$ for any $X \subset A$ such that $1 < |X| < |A|$.

The following are examples of a minimal 1-component in the case $\alpha = 2/3$.
In the rest of the paper, we fix $\alpha = 2/3$ and δ_α will be written δ.



A



B

Let S_A be the set of connected substructures of (A, a, b), i.e., the connected substructures of A containing a and b. Let S_B be the set of connected substructures of (B, a, b). Let $S_0 = S_A \cup S_B$.

Let S_1 be the smallest class with thrifty amalgamation containing S_0.

Lemma 2.2. (1) *If* $(X, a, b) \in S_0$*, then* (X, a, b) *is* (A, a, b)*,* (B, a, b)*, or* $(Y, a, b) \le (X, a, b)$ *for some proper substructure* (Y, a, b) *of* (B, a, b)*.*

(2) *If* $(X, a, b) \in S_0$ *with* $1 < \delta(X) < 2$ *then* $\delta(X) = 4/3$ *or* $5/3$ *and there is* $(Y, a, b) \in S_0$ *such that* $X \le Y$ *and* $\delta(Y) \ge 2$.

Definition 2.3. Let S be a class of structures (X, a, b) where X is a graph and a, b are two distinguished points in X.

Suppose that there are graphs A_1, A_2, …, A_n and points $a_{i-1}, a_i \in A_i$ such that (A_i, a_{i-1}, a_i) is isomorphic to some element of S for each i, and

$$Y = A_1 \oplus_{a_1} A_2 \oplus_{a_2} \cdots \oplus_{a_{n-1}} A_n.$$

We call Y a *S-chain.* n is called the length of the S-chain Y. Each A_i is called an *amalgamand* of Y. With such Y, if we can write

$$X = Y/(a_0 = a_n)$$

then we call X a S-cycle. n is called the length of the S-cycle X. Each amalgamand of Y is also called an *amalgamand* of X.

If S consists of one graph with two points and one edge, we simply call an S-chain a chain, and an S-cycle a cycle.

Let $\mathbf{K}_0$ be the set of S_1-cycles of length greater than $|B|$.

Proposition 2.4. *Suppose* $X \in K_0$.

(1) $\delta(X) = 0$ *if and only if every amalgamand of* X *is isomorphic to* (A, a, b) *or* (B, a, b).
(2) $\delta(X) = 1/3$ *if and only if exactly one amalgamand of* X *is isomorphic to a proper substructure of* (A, a, b) *or* (B, a, b) *with* $\delta = 4/3$ *and each of the remaining amalgamands is isomorphic to* (A, a, b) *or* (B, a, b).
(3) $\delta(X) = 2/3$ *if and only if either exactly one amalgamand of* X *is isomorphic to a proper substructure of* (A, a, b) *or* (B, a, b) *with* $\delta = 5/3$ *or exactly two amalgamands of* X *are isomorphic to a proper substructure of* (A, a, b) *or* (B, a, b) *with* $\delta = 4/3$, *and each of the remaining amalgamands is isomorphic to* (A, a, b) *or* (B, a, b).
(4) $0 < \delta(X) < 1$ *if and only if* $\delta(X) = 1/3$ *or* $\delta(X) = 2/3$.

Proposition 2.5. *Suppose* $X \in K_0$.

(1) *If* $\delta(X) = 0$ *then there is no proper substructure of* X *closed in* X.
(2) *If* $\delta(C) \geq 2$ *for exactly one amalgamand* C *of* X, *and each of the remaining amalgamands of* X *is isomorphic to* (A, a, b) *or* (B, a, b), *then there is a closed point of* X *in* C, *and all the closed points of* X *are in* C.
(3) *If* $\delta(C), \delta(D) \geq 2$ *for exactly two amalgamands* C, D *of* X, *and each of the remaining amalgamands of* X *is isomorphic to* (A, a, b) *or* (B, a, b), *then there is a closed point of* X *in* C, *and also in* D, *and all the closed points of* X *are in* C *or* D.

Let K_1 be the set of S_1-chains and its substructures.

Let K_2 be the smallest set with thrifty amalgamation containing K_0 and K_1.

Proposition 2.6. *Suppose* $X \in \mathbf{K}_2$ *and* X *is connected. If* $\delta(X) < 1$ *then* $X \in \mathbf{K}_0$.

Proposition 2.7. *Suppose* c_1 *and* c_2 *are two closed points in* $X \in \mathbf{K}_2$. *Then there is* $Y \in \mathbf{K}_2$ *such that* $X \leq Y$ *and* c_1 *and* c_2 *are connected in* Y.

Proof. If c_1 and c_2 are connected then there is nothing to prove. Suppose c_1 and c_2 are not connected in $X \in \mathbf{K}_2$. Let X_1 be the connected component of X containing c_1 and X_2 the connected component of X containing c_2. If $c_1, c_2 \in U \subset X$, then

$$\delta(U) \geq \delta(U \cap X_1) + \delta(U \cap X_2) \geq 1 + 1 = 2$$

since $c_i \leq U \cap X_i$ for $i = 1, 2$. Hence, $\{c_1, c_2\} \leq X$. Consider a chain C_3 of length 3 with end points c_1 and c_2. then $\{c_1, c_2\} \leq C_3 \in \mathbf{K}_2$. Hence there is $Y \in K_2$ such that X and C_3 are strongly embedded in Y over $\{c_1, c_2\}$. $\square$

3. An Unsaturated Generic Structure

Let M be the generic structure of $\mathbf{K}_2$.

Proposition 3.1. *M has only one connected component with closed points. The other connected components are exactly $\{A, B\}$-cycles.*

Proposition 3.2. *$Th(M)$ is not ω-stable.*

Proof. In a saturated model of $Th(M)$, we have all $\{A, B\}$-chains of countable length by compactness. Therefore, there are continuumly many types over $\emptyset$. □

We will discuss superstability of $Th(M)$ in a sequel paper.

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